3.10. Duality Revisited

1. Connective Duality Revisited. With an array of new connectives in hand, we return to the topic of duality, first explored in terms of the simpler language of Chapter Two. And as in that earlier discussion, we pair connectives as duals guided by the semantic duality of truth tables brought with the True/False Swap. The list of connectives from the previous reading clearly line up as duals, as their truth tables are semantic duals.

$\top$	$(P \mid Q)$	$(P \to Q)$	$(Q \to P)$	$(P \vee Q)$	$(P \leftrightarrow Q)$	$\sim P$	P	Q	$\sim Q$	$(P \oplus Q)$	$(P \wedge Q)$	$(P \,\%\, Q)$	$(Q \,\%\, P)$	$(P \downarrow Q)$	$\bot$
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**
1	0	1	1	1	1	0	1	1	0	0	1	0	0	0	0
1	1	0	1	1	0	0	1	0	1	1	0	1	0	0	0
1	1	1	0	1	0	1	0	1	0	1	0	0	1	0	0
1	1	1	1	0	1	1	0	0	1	0	0	0	0	1	0

Connective duality is thus guided by the following table of duals.

$\vee$	$\wedge$
$\to$	$\%$
$\leftrightarrow$	$\oplus$
$\mid$	$\downarrow$
$\top$	$\bot$

$\sim$

A general point about connective duality is now more noticeable than in Chapter Two, where we had fewer connectives to serve as examples: a number of features of connectives are preserved under duality. For instance, **wedge and vel** enjoy the feature of **commutativity**: the **order of** the two **parts** has no effect on the truth or falsehood of the whole sentence.[1]

$$(P \wedge Q) \equiv (Q \wedge P)$$
$$(P \vee Q) \equiv (Q \vee P)$$

Of course wedge and vel are connective duals. And both the **bicon and** its connective dual, **exor**, **are** likewise **commutative**.

But the **arrow isn't commutative**: it's a familiar point to us by now that ,e.g., "$(P \rightarrow Q)$" isn't logically equivalent to its converse, "$(Q \rightarrow P)$".

P	**Q**	**(P → Q)**	**(Q → P)**
1	1	**1**	**1**
1	0	**0**	**1**
0	1	**1**	**0**
0	0	**1**	**1**

And the connective dual of the arrow, **wo, isn't commutative**.

P	**Q**	**(P % Q)**	**(Q % P)**
1	1	**0**	**0**
1	0	**1**	**0**
0	1	**0**	**1**
0	0	**0**	**0**

Understandably: "P without Q" doesn't mean the same as "Q without P".

[1] Recall that ● ≡ ▲ means: ● and ▲ are logically equivalent. See 2.17.

(Likewise, we can't be casual about the order of parts even when moving from a conditional to its dual: as the True/False Swap illustrates, the dual of "(P → Q)" isn't "(P % Q)," but "(Q % P)".

P	**Q**	**(P → Q)**	**(Q % P)**
1	1	1	0
1	0	0	0
0	**1**	1	**1**
0	0	1	0

P	**Q**	**T/F Dual of « (P → Q) »**
F	F	F
F	**T**	**T**
T	F	F
T	T	F

When taking the dual of a conditional, we (i) replace the arrow with its connective dual, and (ii) switch the order of the two parts.[2])

In the same way, a connective is **associative** (that is: **grouping** of parts doesn't affect its truth or falsehood) if and only if the dual of that connective is also associative. For instance, **wedge and vel are associative**, since grouping of parts makes no difference to the truth value of a 'triple-barreled' conjunction or disjunction.

$$((P \wedge Q) \wedge R) \;\equiv\; (P \wedge (Q \wedge R))$$
$$((P \vee Q) \vee R) \;\equiv\; (P \vee (Q \vee R))$$

But **neither arrow nor wo are associative**.

$$((P \to Q) \to R) \;\not\equiv\; (P \to (Q \to R))$$
$$((P \,\%\, Q) \,\%\, R) \;\not\equiv\; (P \,\%\, (Q \,\%\, R))$$

And one connective is **idempotent** if and only if its dual connective is.

$$(P \wedge P) \;\equiv\; P$$
$$(P \vee P) \;\equiv\; P$$

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[2] This is less surprising if we recall that an argument has a corresponding conditional, its '**leading principle**' (3.X), whose antecedent is the premise(s) of the argument and whose consequent is the conclusion of that argument; and in taking the dual of an argument we switch premise(s) and conclusion (2.34 §1).

Finally, one connective, C_1, **distributes over** a second connective, C_2, if and only if the dual of C_1 distributes over the dual of C_2. Most obviously, **wedge distributes over vel** – and, by duality, **vel distributes over wedge.**.

$$(\underline{P \land} (Q \lor R)) \equiv ((\underline{P \land} Q) \lor (\underline{P \land} R))$$
$$(\underline{P \lor} (Q \land R)) \equiv ((\underline{P \lor} Q) \land (\underline{P \lor} R))$$

As a second example: **vel distributes over arrow**, since the following pair of sentences are logically equivalent.

$$(\underline{P \lor} (Q \to R)) \equiv ((\underline{P \lor} Q) \to (\underline{P \lor} R))$$

But **wedge doesn't distribute over arrow**; for the following pair of sentences are not logically equivalent.

$$(\underline{P \land} (Q \to R)) \not\equiv ((\underline{P \land} Q) \to (\underline{P \land} R))$$

In a situation **where "P" is false, it's true that "$((P \land Q) \to (P \land R))$"** (because false antecedent yields true conditional) **but false that "$(P \land (Q \to R))$"** (since a false left part makes the whole conjunction false).

But with those two results in hand, we immediately have two parallel points about wo, the dual of the arrow. From the earlier equivalence we see, by duality, that **wedge distributes over wo**.[3]

$$(\underline{P \lor} (Q \to R)) \equiv ((\underline{P \lor} Q) \to (\underline{P \lor} R))$$
$$(\underline{P \land} (R \mathbin{\%} Q)) \equiv ((\underline{P \land} R) \mathbin{\%} (\underline{P \land} Q))$$

And just as wedge doesn't distribute over arrow, **vel doesn't distribute over wo**.

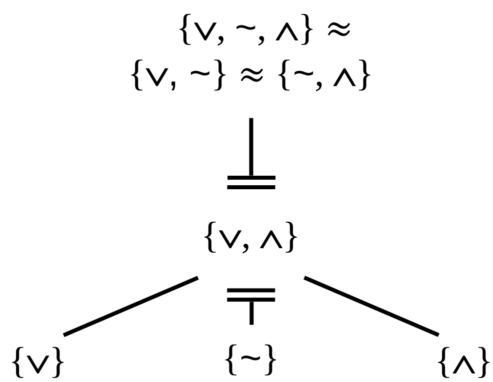
$$(\underline{P \land} (Q \to R)) \not\equiv ((\underline{P \land} Q) \to (\underline{P \land} R))$$
$$(\underline{P \lor} (R \mathbin{\%} Q)) \not\equiv ((\underline{P \lor} R) \mathbin{\%} (\underline{P \lor} Q))$$

[3] Recall that when taking the dual of a conditional or wo sentence, we switch the order of the two parts. So, for example, the dual of "$(Q \to R)$" is "$(R \mathbin{\%} Q)$".

Even the situation proving this point is the dual of the earlier situation: **in a situation where "P" is true, it's false that "((P ∨ R) % (P ∨ Q))" but true that "(P ∨ (R % Q))"**.[4]

2. Duals of Formal Languages Revisited. Treating formal languages as sets of connectives, our new array of connectives yielded a new array of formal languages. And here, as before, we can take the dual of a set of connectives as just the set of the dual of each of those connectives. So the dual of the language {→, ⊥} is the language {%, ⊤}, whereas the language {↔, ⊕} is a self-dual.

Indeed, though the point wasn't remarked upon earlier, each diagram of formal languages wears its duality on its sleeve. That was true already in our Chapter Two 'map' of the formal languages, where each language finds its dual on the opposite side, and the self-dual languages lie along the center 'line of reflection'.



The same holds for the list of expressively adequate, non-redundant formal languages explored in Chapter Three.

[4] As one more application of duality to this case, note that "((P ∧ Q) → (P ∧ R))" is logically equivalent to the anti-valuation sentence "~P ∨ ~Q ∨ R"; whereas its dual "((P ∨ R) % (P ∨ Q))" is logically equivalent to the valuation sentence "~P ∧ ~Q ∧ R".

$\{\downarrow\}$ $\{\mid\}$
$\{\rightarrow, \%\}$
$\{\rightarrow, \sim\}$ $\{\sim, \%\}$
$\{\rightarrow, \oplus\}$ $\{\leftrightarrow, \%\}$
$\{\rightarrow, \bot\}$ $\{\top, \%\}$
$\{\vee, \sim\}$ $\{\sim, \wedge\}$
$\{\vee, \leftrightarrow, \oplus\}$ $\{\leftrightarrow, \oplus, \wedge\}$
$\{\vee, \top, \oplus\}$ $\{\leftrightarrow, \bot, \wedge\}$
$\{\wedge, \top, \oplus\}$ $\{\leftrightarrow, \bot, \vee\}$

And though it wasn’t obvious in Chapter Two, because we didn’t have so many formal languages on offer at that point, this last list points out that expressive adequacy (or inadequacy) is preserved under duality.

> **A formal language is expressively adequate if and only if its dual language is expressively adequate.**

For instance, since $\{\rightarrow, \bot\}$ is expressively adequate, so is its dual $\{\%, \top\}$. And since $\{\leftrightarrow, \oplus\}$ is expressively inadequate, so is its dual (just $\{\leftrightarrow, \oplus\}$ again).

There’s a second moral about duality in the above list, illustrated by the fact that all these languages are non-redundant, and each finds its dual in this same list. As it happens, that’s no coincidence – for the redundancy (or non-redundancy) of a language is also preserved under duality.

> **A formal language is redundant if and only if its dual language is redundant.**

In fact we can sharpen that claim, keeping in mind that for a formal language to be redundant is for that language to have at least one redundant connective – a connective whose removal would not impair the expressive power of the language.

> **If a formal language L_1 is redundant, then it contains some connective that's redundant – call that connective C_1. In that case $D(L_1)$, the dual of language L_1, is also redundant; and $D(L_1)$ contains the redundant connective $D(C_1)$, the dual of connective C_1.**

For example, the language {→, ∨, ⊤} is redundant. Since the arrow already covers the truth tables for vel and tee, those latter two connectives are redundant in that language. And that means the dual language {%, ∧, ⊥} is redundant, and wedge and eet are both redundant connectives in this language. Likewise the language {∨, ~, ∧} is a self-dual, and from the fact that vel is redundant in the language we know that the wedge is also redundant in this language.

All these points are repeated, writ large, in our final 'map' of the super-language **A** and its various sub-languages (repeated on the next page). For once again the languages arrange themselves around a center vertical line, the axis of reflection. And recall a point we made in the previous reading, when first exploring **A**: of all its 5-connective sub-languages, only three are expressively inadequate.

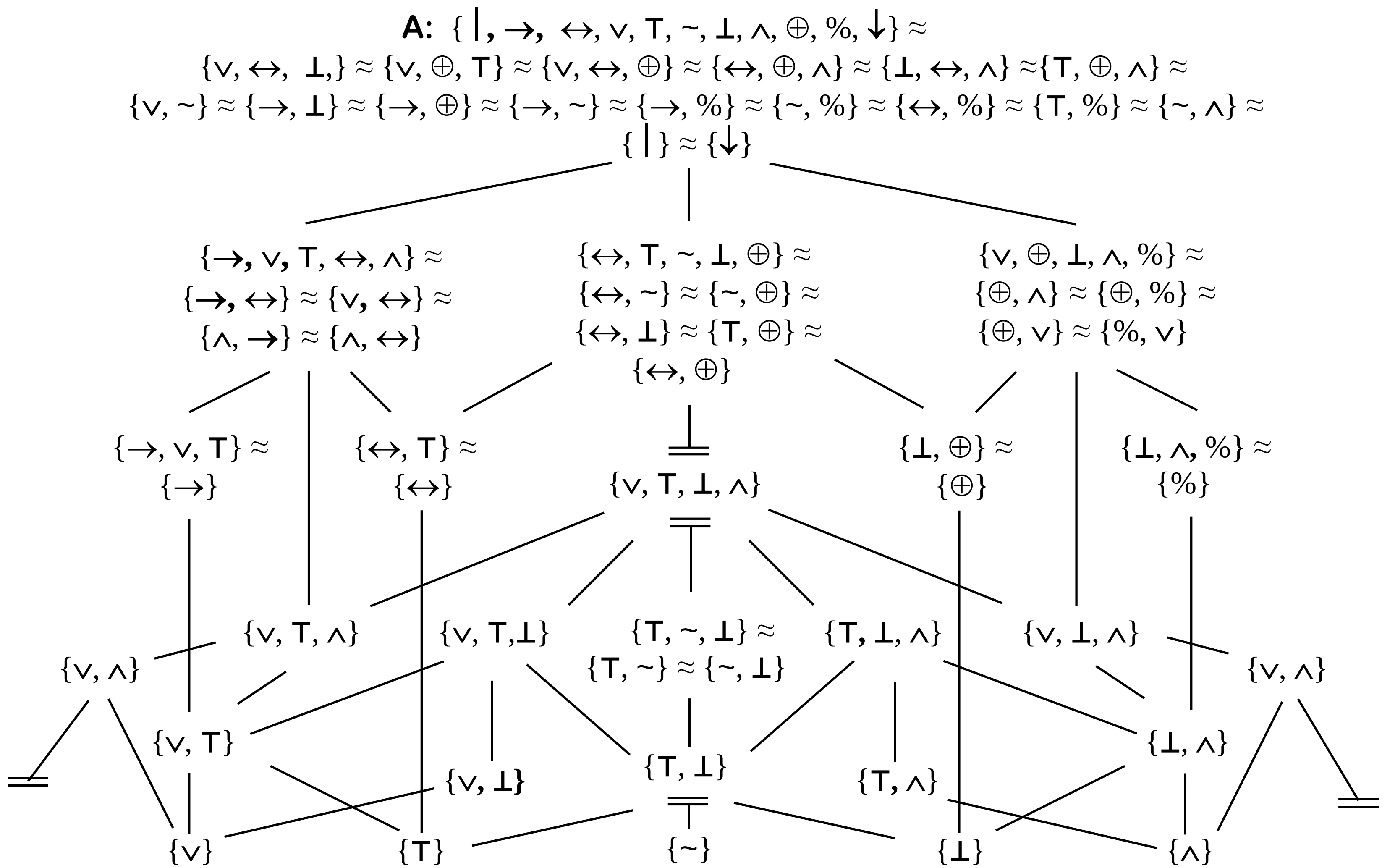
{→, ∨, ⊤, ↔, ∧} {↔, ⊤, ~, ⊥, ⊕} {∨, ⊕, ⊥, ∧, %}

Note that {→, ∨, ⊤, ↔, ∧} is the connective dual of {∨, ⊕, ⊥, ∧, %}, while {↔, ⊤, ~, ⊥, ⊕} is a self-dual.

We remarked that **every 5-connective** sub-language of **A** is **redundant** – an arrangement respecting our principle that the dual of a redundant language is itself a redundant language.

We noted moreover that every **4-connective** sub-language of **A** is **redundant** except {∨, ⊤, ⊥, ∧}. And {∨, ⊤, ⊥, ∧} manages to be the only non-redundant 4-connective sub-language, because {∨, ⊤, ⊥, ∧} is a self-dual.

Finally, every sub-language of **A** with **6 or more connectives** is **expressively adequate** and **redundant** – and each has as its connective dual another language with the same number of connectives, and a like expressive adequacy and redundancy.



A and Its (Main) Sub-Languages

Summary

The following pairs of connectives are connective duals. (The tilde is a self-dual.)

$\lor$	$\land$
$\rightarrow$	%
$\leftrightarrow$	$\oplus$
$\mid$	$\downarrow$
$\top$	$\bot$

$\sim$

A **connective** has any of the following features if and only if its connective dual also has that feature.

- **Commutativity** (order of parts doesn't affect truth/falsehood)
- **Associativity** (grouping doesn't affect truth/falsehood)
- **Idempotence** (putting the same sentence, S, on both sides of the connective yields a sentence logically equivalent to S)

Moreover,

- In language L_1 one connective C_1 **distributes** over a second C_2 if and only if, in the dual language $D(L_1)$, dual $D(C_1)$ distributes over dual $D(C_2)$.

Formal languages have the following duality-governed features.

- A formal language is **expressively adequate** if and only if its dual language is also expressively adequate.

- A formal language is **redundant** if and only if its dual language is redundant.

Moreover,

- If redundant language L_1 has **redundant connective** C_1, dual language $D(L_1)$ is redundant, and has dual connective $D(C_1)$ as a redundant connective.